A short note about IVT

2017年1月30日 15:41

Theorems

- (IVT) If f is continuous on $[a, b]$, $f(a) \neq f(b)$, and N is between $f(a)$ and $f(b)$ then there exists $c \in (a, b)$ that satisfies $f(c) = N$.

This is the version you will see in the exam formula sheet.

Two key points for the theorem are:

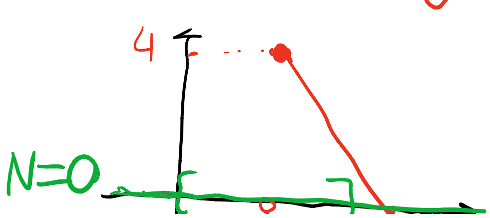
- f is continuous on $[a, b]$
- N is an intermediate value between $f(a)$ and $f(b)$

① (Example where IVT is not applicable)

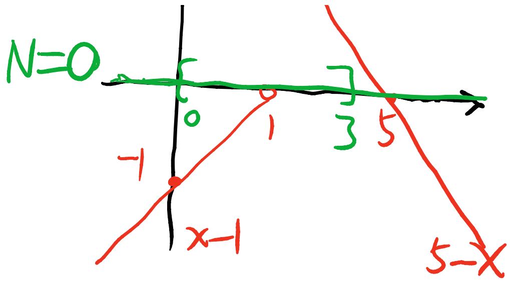
(6 points) Can the Intermediate Value Theorem be applied to $f(x) = \begin{cases} x-1 & \text{if } x < 1 \\ 5-x & \text{if } x \geq 1 \end{cases}$ to show that f has a root on the interval $[0, 3]$? Explain!

Solution: We cannot apply IVT to $f(x)$ since it is not continuous at $x=1$.

Remark. If you draw the picture of $f(x)$, you will see why IVT may fail. Actually,



Since root means $f(x)=0$,



... that means $f(x)=0$,
i.e., we want to apply IVT with $N=0$.

The function has a jump at $1 \in [0,3]$
So the horizontal line $N=0$ may not intersect with the curve.

② (Example where IVT is applicable)

Part 1: Consider the function

$$f(x) = \sin x - x^2 + 1 \quad \text{on} \quad [0,2]$$

Let $N=0$. Verify this $f(x)$ and N satisfy the conditions required by IVT and state the conclusion of IVT.

Solution: $f(x) = \sin x - x^2 + 1$, $\underset{\substack{\uparrow \\ a}}{[0},\underset{\substack{\uparrow \\ b}}{2]}$, $N=0$.

- $f(x)$ is continuous on $[0, 2]$

- $f(0) = \sin 0 - 0^2 + 1 = 1$

  $f(2) = \sin 2 - 2^2 + 1 = \sin 2 - 3 < 0$

  (since $\sin\theta < 1$ for any θ)

  Compare $N=0$ with $f(0)$ and $f(2)$

  We have

$$\underset{\substack{\uparrow \\ 1}}{f(0)} > \underset{\substack{\uparrow \\ 0}}{N} > \underset{\substack{\uparrow \\ \sin 2 - 3}}{f(2)}.$$

N is indeed an intermediate value between $f(0)$ and $f(2)$.

Therefore, the requirements of IVT are fullfilled. IVT is applicable with this f, N and a, b.

And the conclusion is:

there exists $c \in (0, 2)$ such that

there exists $c \in (\underset{a}{0}, \underset{b}{2})$ such that

$$f(c) = N, \text{ i.e.,}$$

$$\sin c - c^2 + 1 = 0. \qquad \blacksquare$$

Part 2, use Part 1 to prove that the equation $\sin x = x^2 - 1$ has at least one root.

Solution: Write $f(x) = \sin x - (x^2 - 1)$

Equation $\sin x = x^2 - 1$ has one root is equivalent to

$\sin x - (x^2 - 1) = 0$ has one root,

ie, equivalent to

$f(x) = 0$ has one root

i.e. equivalent to there is c

i.e., equivalent to there is c such that $f(c)=0$

And this is just what we proved in Part 1. ∎

Remark. In real exam, you will not see Part 1. The problem will be given in the following form:

(6 points) Use the IVT to show that the equation $\sin x = x^2 - 1$ has at least one root.
(*Remember to state why you can apply the IVT*)

The hard part is how to convert the problem into how we state it in Part 1.

The key step is to carefully choose your function $f(x)$

[illegible] intermediate value [illegible] your intermediate value N.

and the suitable interval $[a, b]$

such that IVT is applicable.